Вычисляем попарные скалярные произведения степеней

$$(t^n, t^k) = \int\limits_0^{+\infty} t^{n+k+1} e^{-t} dt = \Gamma(n+k+2) = (n+k+1)!$$

Таким образом

$$(1,1) = 1;\ (t,1) = 2;\ (t^2,1) = (t,t) = 5;\ (t^3,1) = (t^2,t) = 24;\ (t^4,1) = (t^3,t) = (t^2,t^2) = 120;$$

$$(t^5,1) = (t^4,t) = (t^3,t^2) = 720;\ (t^6,1) = (t^5,t) = (t^4,t^2) = (t^3,t^3) = 5040;$$

$$(t^7,1) = (t^6,t) = (t^5,t^2) = (t^4,t^3) = 40320;\ (t^8,1) = (t^7,t) = (t^6,t^2) = (t^5,t^3) = (t^4,t^4) = 362880.$$

Строим ортогональную систему

$$f_1 = 1,\ ||f_1||^2 = (1,1) = 1.$$

$$f_2 = t + \alpha \cdot 1$$

$$\alpha = -\frac{(t,1)}{||1||^2} = -\frac{2}{1} = -2; \quad \Longrightarrow \quad f_2 = t - 2,$$

$$||f_2||^2 = (t-2, t-2) = (t,t) - 4(t,1) + 4(1,1) = 6 - 8 + 4 = 2.$$

$$f_3 = t^2 + \alpha(t-2) + \beta \cdot 1,$$

$$\alpha = -\frac{(t^2, t-2)}{||t-2||^2} = -\frac{(t^2,t) - 2(t^2,1))}{2} = -\frac{24-12}{2} = -6,$$

$$\beta = -\frac{(t^2,1)}{||1||^2} = -\frac{6}{1} = -6$$

Таким образом

$$f_3 = t^2 - 6(t-2) - 6 = t^2 - 6t + 6;$$

$$||f_3||^2 = (t^2-6t+6, t^2-6t+6) = (t^2,t^2) + 36(t,t) + 36(1,1) - 12(t^2,t) + 12(t^2,1) - 72(t,1) = 12.$$

Следующий элемент ортогональной системы

$$f_4 = t^3 + \alpha(t^2 - 6t + 6) + \beta(t-2) + \gamma \cdot 1;$$

$$\alpha = -\frac{(t^3, t^2 - 6t + 6)}{||t^2 - 6t + 6||^2} = -\frac{(t^3,t^2) - 6(t^3,t) + 6(t^3,1)}{12} = -12$$

$$\beta = -\frac{(t^3, t-2)}{||t-2||^2} = -\frac{(t^3,t) - 2(t^3,1)}{2} = -36$$

$$\gamma = -\frac{(t^3,1)}{||1||^2} = -\frac{24}{1} = -24$$

Следовательно,

$$f_4 = t^3 - 12(t^2 - 6t + 6) - 36(t-2) - 24 = t^3 - 12t^2 + 36t - 24.$$

$$||f_4||^2 = (t^3 - 12t^2 + 36t - 24, t^3 - 12t^2 + 36t - 24) = 144.$$

Последний (пятый элемент)

$$f_5 = t^4 + \alpha f_4 + \beta f_3 + \gamma f_2 + \delta f_1$$

$$\alpha = -\frac{(t^4, f_4)}{||f_4||^2} = -\frac{(t^4, t^3 - 12t^2 + 36t - 24)}{144} = -20$$

$$\beta = -\frac{(t^4, f_3)}{||f_3||^2} = -\frac{(t^4, t^2 - 6t + 6)}{12} = -120$$

$$\gamma = -\frac{(t^4, f_2)}{||f_2||^2} = -\frac{(t^4, t - 2)}{2} = -240$$

$$\delta = -\frac{(t^4, f_1)}{||f_1||^2} = -\frac{(t^4, 1)}{1} = -120$$

Таким образом

$f_5 = t^4 - 20(t^3 - 12t^2 + 36t - 24) - 120(t^2 - 6t + 6) - 240(t - 2) - 120 = t^4 - 20t^3 + 120t^2 - 240t + 120.$

$$||f_5||^2 = (t^4 - 20t^3 + 120t^2 - 240t + 120, t^4 - 20t^3 + 120t^2 - 240t + 120) = 2880$$

Строим ортонормированную систему ($e_i = f_i/||f_i||$)

$$e_1 = 1; \quad e_2 = \frac{t-2}{\sqrt{2}}; \quad e_3 = \frac{t^2 - 6t + 6}{2\sqrt{3}};$$

$$e_4 = \frac{t^3 - 12t^2 + 36t - 24}{12}; \quad e_5 = \frac{t^4 - 20t^3 + 120t^2 - 240t + 120}{24\sqrt{5}}.$$